On some Classical Varieties and Codes

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Abstract: We have studied the linear codes associated: a) with the surfaces V_2^{r-1} of order $r-1$ of $PG(r,q)$, analyzing in detail the case $r=4$; b) with the Schubert subvarieties of the Grassmannian varieties.

1 Introduction

The purpose of this paper is to read classical varieties as linear codes, adopting a geometric point of view. It was introduced first by Goppa with his AG-codes, while studying a generalization of RS-codes and BCH-codes and classical Goppa-codes. Many authors (among them Hirschfeld, van Lint, van der Geer, Tsfasman, Vladut, Nogin) followed the initial study by Goppa. The main goal is to build codes by considering the projective systems arising from the rational points of a variety.

In this work we first study the code C associated to a variety V_2^{r-1} of the r-dimensional projective geometry $PG(r,q)$. The $[n,k]_q$-code C has as basic parameters $n=(q+1)^2$, $k=r+1$, and distances $d_1=q^2+(1-m)q$ (where m is the order of the minimum directrix of V_2^{r-1}), $d_{r-1}=q^2+q$. We analyze then our case $r=4$. After having shown some geometric properties of a V_2^3 with respect to the intersection with hyperplanes, planes and lines, the weigths distribution of the associated code C is found.

In the last part, with the contribution of L. Guerra, we describe the linear codes associated to the Schubert varieties. In the projective space P^m consider a flag of subspaces $A_0 \subset \cdots \subset A_d$ of increasing dimensions $a_0 < \cdots < a_d$ with $a_i \geq i$. Inside the Grassmann variety $G(d,m)$ is the Schubert subvariety Ω, whose points correspond to the d-subspaces L such that dim $L \cap A_i \geq i$ for all i. In [3] Ghorpade and Lachaud proved that if $d(\Omega)$ denotes the dimension of Ω then the minimum distance satisfies $d_1 \leq q^{d(\Omega)}$, and they also conjectured that the equality holds. Here the conjecture is checked for the unique non-trivial Schubert variety in the Klein quadric $G(1,3)$, which is for $(a_0, a_1) = (1,3)$. Moreover we provide a formula for the number of the points of a Schubert variety, and we obtain a lower bound for the minimum distance d_1.

Finally we mention that in a recent preprint by Ghorpade and Tsfasman [?] more results may be found on the basic parameters of the Schubert codes, together with an updated overview of the research on the present subject.

2 Codes and projective systems

Let $F = GF(q)$ be a finite field, $q = p^s$, p prime, denote by F^n the n-dimensional vector space over F.

We begin by briefly recalling some basic definitions.

A **linear** $[n,k]_q$**-code** C is a k-dimensional subspace of F^n. The **dual code** of C is the $(n-k)$-dimensional subspace $C^\perp$ of F^n and it is an $[n, n-k]_q$-code.

For $t \geq 1$ the t-th **higher weight** of C (see Wei [13]) is defined by

$$d_t = d_t(C) = \min\{\|D\| \text{ for all } D < C, \ \dim D = t\},$$

where $\|D\|$ is the number of indices i such that there exists $v \in D$ with $v_i \neq 0$. Note that $d_1 = d_1(C)$ is the classical minimum distance of C, the *Hamming distance*.

The code C (or, $C^\perp$) is of **genus at most** $g \geq 0$ if the following inequalities hold:

$$k + d_1 \geq n + 1 - g \quad \text{and} \quad (n-k) + d_1^\perp \geq n + 1 - g$$

where $d_1^\perp$ is the distance of $C^\perp$. C is an MDS-code when $g = 0$ (see [10]).

Let $P^{k-1} = PrF^k$ denote the $(k-1)$-dimensional Galois projective space $PG(k-1, q)$ over the field F. An $[n,k]_q$-**projective system** $\mathcal{X}$ of P^{k-1} is a collection of n not necessarily distinct points. It is called *non-degenerate* if these n points are not contained in any hyperplane.

Assume that $\mathcal{X}$ consists of n distinct points having rank k. For each point of $\mathcal{X}$ choose a generating vector. Denote by M the matrix having as rows such n vectors and let C be the linear code having M^t as a generator matrix. The code C is the k-dimensional subspace of F^n which is the image of the mapping from the dual k-dimensional space $(F^k)^*$ onto F^n that calculates every linear form over the points of $\mathcal{X}$. Hence the length n of codeword of C is the cardinality of $\mathcal{X}$, the dimension of C being just k. There exists a natural 1–1 correspondence between the equivalence classes of a non-degenerate $[n,k]_q$-*projective system* $\mathcal{X}$ and a non-degenerate $[n,k]_q$-*code* C such that if $\mathcal{X}$ is an $[n,k]_q$-projective system and C is the corresponding code, then the non-zero codewords of C correspond to hyperplanes of P^{k-1}, up to a non-zero factor, the correspondence preserving the parameters n, k, d_t. More generally, subcodes D of C of dimension r correspond to (projective) subspaces of codimension r of P^{k-1}. Consequently, the higher weights of C are given by $d_t = d_t(C) = n - \max\{|\mathcal{X} \cap S| : S < P^{k-1},\ \operatorname{codim} S = t\}$. In particular, $d_1 = d_1(C) = n - \max\{|\mathcal{X} \cap H| : H < P^{k-1},\ \operatorname{codim} H = 1\}$.

The **spectrum** of a projective system $\mathcal{X}$ of P^{k-1} (or of the corresponding linear code C) is defined by the numbers $A_i^{(s)} = |\{S < P^{k-1} : \operatorname{codim} S = s,\ |S \cap \mathcal{X}| = n - i\}| \quad for \quad all \quad i = 1, 2, ..., n, \quad s = 1, 2, ..., k-2$.

For the above definitions see [10] and [3]. The following theorem holds (see [10]):

Theorem 1 *A (non-degenerate) projective system of P^k (or, of the corresponding code with $d_1 \geq 1$) satisfies the following bounds:*

$$\begin{array}{ll} & 1 \leq d_1 \leq d_2 \leq \cdots \leq d_k = n; \\ \textit{for } t \leq s \leq k, & d_t \geq n - \lfloor (q^{k-t}-1)(n-d_s)/q^{k-s} \rfloor; \\ \textit{Singleton-type bound:} & t \leq d_t \leq n - k + t; \\ \textit{Plotkin-type bound:} & d_t \leq \lfloor n(q^t-1)q^{k-t}/(q^k-1) \rfloor; \\ \textit{Griesmer bound:} & n \geq \sum_i \lceil d_1/q^i \rceil, \quad i = 0, \ldots, k-1; \\ \textit{Griesmer-type bound:} & d_t \geq \sum_i \lceil d_1/q^i \rceil, \quad i = 0, \ldots, t-1. \end{array}$$

Codes can be built from classical algebraic varieties by considering projective systems arising from their rational points.

For instance, affine and projective spaces can be considered as Reed–Muller codes. We like to recall codes from quadrics (Wan), Hermitian varieties (Hirschfeld, Tsfasman, Vladut), Del Pezzo surfaces (Boguslawski), Grassmannians (Ryan, Nogin, Ghorpade, Lachaud), without mentioning the many other authors who studied algebraic curves from the codes point of view.

3 The ruled surfaces of order $r-1$ of $PG(r,q)$ and their codes

Let $P^r = PG(r,q)$ be the r-dimensional projective geometry over $F = GF(q)$. If we denote by $\overline{F}$ the algebraic closure of F, we can consider P^r as a subgeometry of the geometry $\overline{P}^r$ of the same dimension over $\overline{F}$.

A **variety V_u^v of dimension u and of order v** of P^r is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{P}^r$ defined by a finite set of polynomials of $F[x_0, \ldots, x_r]$.

The following results are well known (see [2]), they can be easily proved also for the finite case, suitably modified.

Theorem 2 *The varieties V_2^{r-1} of P^r are the rational ruled varieties and the Veronese surface if $r=5$.*

Assume $r \neq 5$. Denote by S_t a projective t-dimensional subspace of P^r for $t < r$.

Theorem 3 (i) *V_2^{r-1} is a ruled rational normal surface.*

(ii) *Every irreducible curve C^t of order $t \leq r-1$ of V_2^{r-1} is a rational normal curve and it does exist in an S_t.*

(iii) *V_2^{r-1} can be built by a projectivity between two irreducible directrix curves $C^m \subset S_m$ and $C^{r-m-1} \subset S_{r-m-1}$.*

(iv) *If the minimum order directrix of V_2^{r-1} is of order m, then h generatrix lines are dependent or independent according to $h \leq m+1$ or $h > m+1$, respectively.*

(v) *If r is even, there exists exactly one directrix curve of order $m = (r-2)/2$; if r is odd, there exist directrix curves of order $m \leq (r-1)/2$.*

Let us consider a surface V_2^{r-1}. Assume that V_2^{r-1} contains a minimum order directrix C^m where $m < q$.

Denote by $\mathcal{X}$ the projective system consisting of the rational points of V_2^{r-1}. It is $|\mathcal{X}| = (q+1)^2$. Let C be the linear code associated to $\mathcal{X}$.

Theorem 4 *C is an $[n,k]_q$-code with*

$$n = (q+1)^2, \quad k = r+1, \quad d_1 = q^2 - mq, \quad d_{r-1} = q^2 + q.$$

Proof. From the previous arguments it follows that $n = (q+1)^2$ and $k = r+1$. To evaluate d_{r-1} simply note that V_2^{r-1} contains lines, therefore $d_{r-1} = q^2+2q+1-(q+1) = q^2+q$. From Theorem 3,(iv)) it follows that the hyperplanes H that give rise to the (minimum) distance d_1, contain the subspace $S_m \supset C^m$ and $m+1$ generatrix lines. Then $|H \cap V_2^{r-1}| = (m+2)q+1$ and therefore $d_1 = q^2+2q+1-((m+2)q+1) = q^2-mq$. To evaluate d_{r-1} simply note that V_2^{r-1} contains lines, therefore $d_{r-1} = q^2+2q+1-(q+1) = q^2+q$. ⋆

Lemma 5 *The inequality $(m+2)q \geq r-1$ holds for every q and r.*

Proof. The inequality follows from $d_1 \leq n-k+1$. ⋆

Theorem 6 *C and $C^\perp$ are of genus at most $g \geq (m+2)q-(r-1)$.*

Proof. It follows from the definition of genus of a code, Theorem 4 and Lemma 5. ⋆

Consider now the case $r = 4$. Let V_2^3 be a ruled surface of $P^4 = PG(4,q)$.

Lemma 7 (a) *The variety V_2^3 has a line l as a minimum order directrix, projecting a non-degenerate conic C of a plane π skew to l.*

(b) *V_2^3 consists of $q+1$ generatrix lines, skew to each other, birationally connecting the points of l and C.*

(c) *There exists only one hyperplane H containing l and such that the line $\pi \cap H$ is skew to l.*

(d) *There exist hyperplanes H' containing one generatrix g_1 and such that $l \not\subset H'$ for which $H' \cap V_2^3 = \{g_1, C^2\}$ for some conic C^2.*

(e) *There exist hyperplanes H' containing two generatrices g_1, g_2 for which $H' \cap V_2^3 = \{g_1, g_2, l\}$, such hyperplanes having the maximum intersection with V_2^3.*

(f) *There are no hyperplanes containing 3 generatrices.*

(g) *If P, Q are points of V_2^3 then either P, Q belong tothe minimum order directrix l, or to a generatrix g_1, or they are points of a unique conic C^2 of V_2^3.*

(h) *There exist planes π' intersecting V_2^3 either in one point, or in one line, or in one irreducible conic, or in two intersecting lines (namely, l and a generatrix g_1). The last planes are tangent and have the maximum cardinality intersection with V_2^3.*

(i) *The varieties V_2^3 of P^4 having l and C as directrices are projectively equivalent and their number is $(q+1)q(q-1)$.*

Proof. See Theorem 3, [12, Proposition 1.1, Lemma 1.2, Theorems 1.2, 1.3], and [1, Propositions 1.1, 1.3, 1.4, 1.5]. ⋆

As above, denote by $\mathcal{X}$ the projective system consisting of the rational points of V and by C the linear code associated to it.

Theorem 8 *C is an $[n, k]_q$-code with $n = (q+1)^2$, $k = 5$, $d_1 = q^2 - q$, $d_2 = q^2$, $d_3 = q^2 + q$. C (and $C^\perp$) is of positive genus $g \geq 3q - 3$.*

Proof. From Theorem 4 follow immediately the expressions of d_1 and d_3. Remark that from Lemma 7, (h)the planes that contribute to evaluate d_2 are those containing the minimum order directrix l and a generatrix g_1, therefore $d_2 = q^2 + 2q + 1 - (2q + 1) = q^2$.

From Theorem 6 and Lemma 7, (a) it follows that $g \geq 3q - 3$. ⋆

Theorem 9 *The spectrum $A_i^{(1)}$ of $\mathcal{X}$ is*

$$A_{d_1}^{(1)} = (q+1)q/2, \quad A_{d_2}^{(1)} = (q^2 - q)(q+1), \quad A_{d_3}^{(1)} = (q^4 + 1) + q(q+3)/2,$$

$$A_i^{(1)} = 0 \quad for \quad all \quad i \in \{1, 2, ..., n\} \setminus \{d_1, d_2, d_3\}.$$

Proof. From Lemma 7 it follows $|H \cap V_2^3| \in \{q+1, 2q+1, 3q+1\}$ for all hyperplanes H, so that from the definition of spectrum we get

$$A_i^{(1)} = 0 \quad for \quad all \quad i \in \{1, 2, ..., n\} \setminus \{d_1, d_2, d_3\}.$$

Remark that $A^{(1)}_{d_1}$ equals the number a of the hyperplanes containing two skew lines of the $q+1$ generatrices of V^3_2 (and then also the minimum order directrix l) and that $A^{(1)}_{d_2}$ equals the number b of the hyperplanes containing one generatrix (and also a conic, from Lemma 7, (d)) of V^3_2. We get first $A^{(1)}_{d_1} = a = (q+1)!/2(q-1)! = (q+1)q/2$.

To compute $A^{(1)}_{d_2}$ remark that the number of all the hyperplanes through the $q+1$ generatrices of V^3_2 is $(q^2+q+1)(q+1) = 2a+b$. Therefore $b = (q^2-q)(q+1) = A^{(1)}_{d_2}$.

$A^{(1)}_{d_3}$ is simply the number of the remaining hyperplanes, that is,

$$A^{(1)}_{d_3} = q^4+q^3+q^2+q+1-(a+b) = (q^4+1)+q(q+3)/2. \quad \star$$

4 Codes from Schubert varieties

The Grassmann codes have been studied in a number of papers [?], [?], [?], [?], and their basic parameters are known. Denote by $V = F^{m+1}$ the $(m+1)$-dimensional vector space over the field $F = GF(q)$. For the grassmannian $G(d,m)$ of d-dimensional subspaces in $P^m = PG(m,q) = PrV$ the associated code has length $n =: \left[\begin{smallmatrix} m+1 \\ d+1 \end{smallmatrix} \right]$ (that is, the Gaussian binomial coefficient), dimension $k = \left(\begin{smallmatrix} m+1 \\ d+1 \end{smallmatrix} \right)$ and weights $d_1 = q^{(m-d)(d+1)}$, $d_t = d_1(q^t-1)/(q^t - q^{t-1})$ for $t > 1$.

In the projective space P^m consider a flag of subspaces $A_0 \subset A_1 \subset \cdots \subset A_d$ of increasing dimensions $a_0 < a_1 < \cdots < a_d$ so that $a_i \geq i$. Inside the Grassmann variety $G(d,m)$ is the **Schubert subvariety** $\Omega(a) = \Omega(a_0, a_1, \ldots, a_d)$, that is the set of the points corresponding to the d-subspaces L such that dim $L \cap A_i \geq i$ for all i. It is irreducible of dimension $a_0 + a_1 + \cdots + a_d - \frac{1}{2}d(d+1)$. The dimension $k(a) = k(a_0, a_1, \ldots, a_d)$ of the associated code $C(\Omega(a))$ is known to coincide with the number of increasing sequences $b_0 < b_1 < \cdots < b_d$ such that $b_i \leq a_i$, cf. [?]. We are going to approach the lenght and the minimum weight.

Example 10 In the Klein quadric $G(1,3)$ the only nontrivial Schubert subvariety is for $(a_0, a_1) = (1,3)$. It is the variety of lines meeting a given line, the *special linear complex of lines*. A direct computation gives the parameters $n = q(q+1)^2 + 1$ and $k = 5$ and weights

$$d_1 = q^3,\ d_2 = q^3 + q^2,\ d_3 = q^3 + 2q^2.$$

The weights may be computed using the quadratic equation $x_0x_5 - x_1x_4 + x_2x_3 = 0$ of the Grassmannian in P^5, and cutting out the Schubert variety by means of the hyperplane $P^4 : x_5 = 0$. The section is a cone with vertex $(1,0,0,0,0)$ over the hyperbolic quadric $x_1x_4 - x_2x_3 = 0$ in P^3. In particular the cone is filled with $q+1$ planes, obtained from the lines of a ruling of the quadric, two of which only meet at the vertex. In order to compute d_1 we want the maximum intersection with hyperplanes in P^4. It is easy to see that this is obtained if the hyperplane contains one of the planes, and meets each of the remaining q planes in a line, which goes through the vertex. Then for d_2 we want the maximum intersection with 2-subspaces in P^4, and this is obtained if the subspace is one of the ruling planes, or is a plane through the vertex which meets every ruling plane in a line. Finally d_3 is clear.

The number of points

Denote by $n(a) = n(a_0, a_1, \ldots, a_d)$ the number of points of the Schubert variety $\Omega(a)$ of P^m which gives the length of the associated Schubert code $C(\Omega(a))$. We present an effective formula for this number of points.

The formula requires the function $\eta(r,s,t)$ which gives, in a vector space of dimension s, the number of r-subspaces L such that $L \cap A = 0$, where A is a given t-subspace. It is well known that

$$\eta(r,s,t) = \begin{bmatrix} s-t \\ r \end{bmatrix} q^{tr}.$$

For a given sequence of dimensions $l_0 \leq \ldots \leq l_d$ such that $a_i \geq l_i \geq i$ and $l_d = d$, define $n(a,l) = n(a_0, \ldots, a_d, l_0, \ldots, l_d)$ to be the number of d-dimensional subspaces L such that $\dim L \cap A_i = l_i$.

Proposition 11 *The number of points of $\Omega(a)$ is given by*

$$n(a) = \sum_l n(a,l),$$

where

$$n(a,l) = \prod_{i=0}^{d} \eta(r_i, s_i, t_i),$$

$$r_i = l_i - l_{i-1}, \quad s_i = a_i - l_{i-1}, \quad t_i = a_{i-1} - l_{i-1},$$

where for $i = 0$ *we mean that* $\eta(r_0, s_0, t_0) := \begin{bmatrix} a_0 + 1 \\ l_0 + 1 \end{bmatrix}$, *which also is a special case of the formula for* η *if we agree that* $a_{-1} = l_{-1} = -1$.

Proof. The first formula is quite clear. For the second formula we observe that there is a one to one correspondence between subspaces L of P^m such that $\dim L \cap A_i = l_i$ and chains of subspaces $L_0 \subset L_1 \subset \cdots \subset L_d$ with $L_i \subset A_i$ and $\dim L_i = l_i$ and such that $L_i \cap A_{i-1} = L_{i-1}$. Then there is a flag of vector subspaces $A'_0 \subset A'_1 \subset \cdots \subset A'_d$ and each L_i in the chain above is projectivized of a vector subspace $L'_i \subset V$. We therefore assume that the chain has been constructed up to L_{i-1}. Then the number of possible choices for L_i is the number of subspaces L'_i/L'_{i-1} in A'_i/L'_{i-1} having zero intersection with A'_{i-1}/L'_{i-1}. $\star$

The minimum weight, a lower bound

The grassmannian $G(d,m)$ of the d-subspaces of $P = \operatorname{Pr} V$ is embedded into $\operatorname{Pr} \wedge^{d+1} V$. If $L \subset P$ is projectivized of $L' \subset V$ then the decomposable $(d+1)$-vector $e_0 \wedge e_1 \wedge \cdots \wedge e_d$ obtained from a basis of L' is determined up to proportionality. Every hyperplane is defined as $(e_0 \wedge e_1 \wedge \cdots \wedge e_d) \wedge \omega = 0$ for some nonzero $(m-d)$-vector $\omega \in \wedge^{m-d} V$, and ω is a word in the code associated to the grassmannian $G(d,m)$.

The Schubert variety $\Omega(a)$ is contained into hyperplanes H_ω for values of ω which make up a linear subspace $\mathcal{L}(a) \subset \wedge^{m-d} V$. The linear closure of $\Omega(a)$ is projectivized of the linear subspace of $\wedge^{d+1} V$ annihilated by $\mathcal{L}(a)$, so every hyperplane in the linear closure is traced by H_ω for some $\omega \in \wedge^{m-d} V$ such that $\omega \notin \mathcal{L}(a)$, and the equivalence class $[\omega] \in \wedge^{m-d} V/\mathcal{L}(a)$ is a word of the Schubert code $C(\Omega(a))$.

We denote by $n(a_0, a_1, \dots, a_d, \omega)$ the number of subspaces which satisfy the Schubert condition and which do not belong to the hyperplane H_ω. This is the weight of the word ω of the Schubert code. We then denote by $d_1(a_0, a_1, \dots, a_d)$ the minimum weight of code words.

Theorem 12 *We have the inequality*

$$d_1(a) \geq \frac{(q^{a_0+1} - q^{a_0})\,(q^{a_1} - q^{a_1-1}) \cdots (q^{a_d-d+1} - q^{a_d-d})}{(q^{d+1}-1)\,(q^d-1)\cdots(q-1)}$$

where the right side is asymptotically $\sim q^{a_0+a_1+\cdots+a_d-d(d+1)}$.

Recall that the inequality

$$d_1(a) \leq q^{a_0+a_1+\cdots+a_d-\frac{1}{2}d(d+1)}$$

is known and that equality here is indeed conjectured [**?**]. The conjecture is verified for $d=1$ and $m=3$ as follows from Example **??**.

Proof. Consider the flag of vector subspaces $A'_0 \subset A'_1 \subset \cdots \subset A'_d$ of dimensions dim $A'_i = a_i + 1$ in the vector space V, and for $\omega \in \wedge^{m-d}V$ define $\eta(a_0, a_1, \ldots, a_d, \omega)$ to be the number of independent sequences of vectors $(e_0, e_1, \ldots, e_d)$ such that $e_0 \in A'_0, e_1 \in A'_1, \ldots, e_d \in A'_d$ and such that $(e_0 \wedge e_1 \wedge \cdots \wedge e_d) \wedge \omega \neq 0$. Then

$$[d+1]\; n(a,\omega) \geq \eta(a,\omega)$$

where $[d+1] = (q^{d+1}-1)\,(q^{d+1}-q)\cdots(q^{d+1}-q^d)$ is the number of bases in a fixed vector space of dimension $d+1$.

In order to estimate $\eta(a,\omega)$ we reason by induction on d. We consider that for a nonzero multivector ω, of arbitrary order, the number of vectors $e_0 \in A'_0$ such that $e_0 \wedge \omega \neq 0$ is $\geq q^{a_0+1} - q^{a_0}$, cf. [**?**]. This implies the statement for $d=0$.

For $d > 0$ we now want to estimate for every admissible e_0 all possible sequences $(e_1, \ldots, e_d)$ modulo e_0. We therefore assume that $e_0 \in A'_0$, such that $e_0 \wedge \omega \neq 0$ is fixed. Write $V = \langle e_0 \rangle \oplus \bar{V}$ for some complementary subspace, so every e_i has a component $\bar{e}_i \in \bar{V}$ and every subspace in the flag is written as $A'_i = \langle e_0 \rangle \oplus \bar{A}_i$ and moreover the multivector ω is written as $\omega = e_0 \wedge \omega_1 + \bar{\omega}$ where $\omega_1 \in \wedge^{m-d-1}\bar{V}$ and $\bar{\omega} \in \wedge^{m-d}\bar{V}$. Then $e_0 \wedge e_1 \wedge \cdots \wedge e_d \wedge \omega = e_0 \wedge \bar{e}_1 \wedge \cdots \wedge \bar{e}_d \wedge \bar{\omega}$ and the product is zero if and only if $\bar{e}_1 \wedge \cdots \wedge \bar{e}_d \wedge \bar{\omega} = 0$.

Consider the number $\eta(a_1 - 1, \ldots, a_d - 1, \bar{\omega})$ of independent sequences $(\bar{e}_1, \ldots, \bar{e}_d)$ in $\bar{V}$ such that $\bar{e}_1 \in \bar{A}_1, \ldots, \bar{e}_d \in \bar{A}_d$ and such that $(\bar{e}_1 \wedge \cdots \wedge \bar{e}_d) \wedge \bar{\omega} \neq 0$. Then using the bound quoted above for the number of admissible vectors e_0 we obtain

$$\eta(a_0, a_1, \ldots, a_d, \omega) \geq (q^{a_0+1} - q^{a_0})\; q^d\; \eta(a_1 - 1, \ldots, a_d - 1, \bar{\omega}).$$

The statement for dimension d is therefore reduced to the statement for dimension $d-1$. $\star$

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